SOLUTION BY OLIVE C. HAZLETT, Bryn Mawr College.

The theorem is clearly true for all primes. Accordingly, assume the theorem is true for all divisors of $n = \prod_{i=1}^{k} p_i^{e_i}$ which are less than n. Now for n the defining equation becomes

$$\phi(n) + A_n + p_1^{e_1}\left(1 - \frac{1}{p_1}\right) B_n = n,$$

where A_n is the sum of the ϕ-functions formed for the divisors of $p_1^{e_1-1}\, p_2^{e_2} \cdots p_k^{e_k}$ and B_n is a similar sum formed for all distinct factors of any of the numbers $p_2^{e_2-1}\, p_3^{e_3} \cdots p_k^{e_k}$, $p_2^{e_2}\, p_3^{e_3-1}\, p_4^{e_4} \cdots p_k^{e_6}$, $\cdots$. It is easy to find an expression for A_n, but it is sufficient for our purposes to note that A_n is a polynomial in $p_2, \cdots, p_k$ of degree at most $\sum_{i=2}^{k} e_i - 1$. Therefore $p_1^{e_1}\left(1 - \frac{1}{p_1}\right)$ is a factor of $\phi(n)$. Since this proof is perfectly general, it holds for every expression of the form $p_i^{e_i}\left(1 - \frac{1}{p_i}\right)$ $(i = 1, \cdots, p)$, and thus $\prod_{i=1}^{k} p_i^{e_i}\left(1 - \frac{1}{p_i}\right)$ is a factor of $\phi(n)$. Comparing the coefficients of $\prod_{i=1}^{k} p_i^{e_i}$ our formula is proved.

Also solved by H. C. FEEMSTER.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

DISCUSSIONS.

I. RELATING TO FINDING DERIVATIVES OF TRIGONOMETRICAL FUNCTIONS.

By T. H. HILDEBRANDT, University of Michigan.

In most textbooks on the elementary calculus the derivatives of the trigonometric functions are based on the derivative of the sine function, which, in turn, is derived from the definition of derivative. The proofs dealing with the value of this derivative seem to have something indirect about them. All goes well until the point is reached where the expression

$$\lim_{\Delta x \doteq 0} \frac{\sin (x + \Delta x) - \sin x}{\Delta x}$$

is to be evaluated, and then one of two methods is used. Either $\sin (x + \Delta x)$ is expanded by the formula for the sine of the sum of two angles and the formula for $1 - \cos x$ in terms of half angles is used, or the formula for the difference of two sines is used. Both of these latter formulæ have long since escaped the memory of the average sophomore student—if they ever had lodging there—and he practically accepts this part of the derivation on faith.

While it must be admitted that the most natural beginning for a chapter on the derivatives of trigonometrical functions is a paragraph devoted to finding the derivative of the sine, this advantage is more than counterbalanced by the simplicity with which it is possible to obtain the derivative of the tangent function directly from the definition of derivative—a fact which seems almost to have

escaped the attention of writers of textbooks on calculus. For this purpose it is possible to proceed in either of two ways, both of which are elegant and altogether natural and direct. If we take $\tan x = \sin x/\cos x$, then

$$\begin{aligned}\tan(x+\Delta x) - \tan x &= \frac{\sin(x+\Delta x)}{\cos(x+\Delta x)} - \frac{\sin x}{\cos x} \\ &= \frac{\sin(x+\Delta x)\cos x - \cos(x+\Delta x)\sin x}{\cos(x+\Delta x)\cos x} \\ &= \frac{\sin\Delta x}{\cos(x+\Delta x)\cos x}.\end{aligned}$$

If we divide now by Δx and take the limit as Δx approaches zero, then $\lim_{\alpha \doteq 0} (\sin\alpha)/\alpha = 1$ is applicable for evaluating the derivative. Or, proceeding directly, we have

$$\begin{aligned}\tan(x+\Delta x) - \tan x &= \frac{\tan x + \tan\Delta x}{1 - \tan x \tan\Delta x} - \tan x \\ &= \frac{(1+\tan^2 x)\tan\Delta x}{1 - \tan x\tan\Delta x}.\end{aligned}$$

Then by using the fact that $\lim_{\alpha \doteq 0} (\tan\alpha)/\alpha = 1$, we again get at once the value of the derivative of the tangent. Either of these methods yields this derivative without any troublesome trigonometrical transformations.

On the basis of the derivative of the tangent, the remaining derivatives are easily obtained in the order $\sec x$, $\cos x$, and $\sin x$: the secant by taking derivatives in the relation

$$\tan^2 x + 1 = \sec^2 x,$$

the cosine through the fact that it is the reciprocal of the secant, and the sine from one of the relations

$$\sin x = \cos(\pi/2 - x), \qquad \sin x = \cos x \tan x.$$

We get, then, in this way the derivatives which are used most, and we employ only trigonometrical relations which are familiar to the average sophomore student.

II. Relating to the Proof that a Rigid Body Moving about a Fixed Point Is at Each Instant Rotating about an Axis through that Point.

By E. L. Rees, University of Kentucky.

The following proof of this theorem differs from other proofs b; methods in that it is based upon an interesting geometric interpretation of the equations involved.